Tensor products of $C(X)$-algebras over $C(X)$

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Abstract

Given a Hausdorff compact space X, we study the C*-(semi)-norms on the algebraic tensor product $A \otimes_{alg,C(X)} B$ of two $C(X)$-algebras A and B over $C(X)$. In particular, if one of the two $C(X)$-algebras defines a continuous field of C*-algebras over X, there exist minimal and maximal C*-norms on $A \otimes_{alg,C(X)} B$ but there does not exist any C*-norm on $A \otimes_{alg,C(X)} B$ in general.

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0 Introduction

Tensor products of C*-algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\min} B$ of two C*-algebras A and B always defines the minimal C*-norm on the algebraic tensor product $A \otimes_{alg} B$ of A and B over the complex field $\mathbb{C}$.

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$-algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$-algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$-algebras framework.

Let us introduce the following definition:

DEFINITION 0.1 *Given two $C(X)$-algebras A and B, we denote by $\mathcal{I}(A,B)$ the involutive ideal of the algebraic tensor product $A \otimes_{alg} B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.*

Our aim in the present article is to study the C*-norms on the algebraic tensor product $(A \otimes_{alg} B)/\mathcal{I}(A,B)$ of two $C(X)$-algebras A and B over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A,B) \subset A \otimes_{alg} B$ which contains $\mathcal{I}(A,B)$ such that every C*-semi-norm on $A \otimes_{alg} B$ which is zero on $\mathcal{I}(A,B)$ is also zero on $\mathcal{J}(A,B)$ and we prove that there always exist a minimal C*-norm $\| \;\|_m$ and a maximal C*-norm $\| \;\|_M$ on the quotient $(A \otimes_{alg} B)/\mathcal{J}(A,B)$.

We then study the following question of G.A. Elliott ([5]): when do the two ideals $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$ coincide?

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1 Preliminaries

We briefly recall here the basic properties of $C(X)$-algebras.

Let X be a Hausdorff compact space and $C(X)$ be the C*-algebra of continuous functions on X. For $x \in X$, define the morphism $e_x : C(X) \to \mathbb{C}$ of evaluation at x and denote by $C_x(X)$ the kernel of this map.

DEFINITION 1.1 *([10]) A $C(X)$-algebra is a C*-algebra A endowed with a unital morphism from $C(X)$ in the center of the multiplier algebra $M(A)$ of A.*

We associate to such an algebra the unital $C(X)$-algebra $\mathcal{A}$ generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, denote by A_x the quotient of A by the closed ideal $C_x(X)A$ and by a_x the image of $a \in A$ in the fibre A_x. Then, as

$$\|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\},$$

the map $x \mapsto \|a_x\|$ is upper semi-continuous for all $a \in A$ ([14]).

Note that the map $A \to \oplus A_x$ is a monomorphism since if $a \in A$, there is a pure state ϕ on A such that $\phi(a^*a) = \|a\|^2$. As the restriction of ϕ to $C(X) \subset M(A)$ is a character, there exists $x \in X$ such that ϕ factors through A_x and so $\phi(a^*a) = \|a_x\|^2$.

Let $S(A)$ be the set of states on A endowed with the weak topology and let $\mathcal{S}_X(A)$ be the subset of states φ whose restriction to $C(X) \subset M(A)$ is a character, i-e such that there exists an $x \in X$ (denoted $x = p(\varphi)$) verifying $\varphi(f) = f(x)$ for all $f \in C(X)$. Then the previous paragraph implies that the set of pure states $P(A)$ on A is included in $\mathcal{S}_X(A)$.

Let us introduce the following notation: if $\mathcal{E}$ is a Hilbert A-module where A is a C*-algebra, we will denote by $\mathcal{L}_A(\mathcal{E})$ or simply $\mathcal{L}(\mathcal{E})$ the set of bounded A-linear operators on $\mathcal{E}$ which admit an adjoint ([9]).

DEFINITION 1.2 *([3]) Let A be a $C(X)$-algebra.*

A $C(X)$-representation of A in the Hilbert $C(X)$-module $\mathcal{E}$ is a morphism $\pi : A \to \mathcal{L}(\mathcal{E})$ which is $C(X)$-linear, i.e. such that for every $x \in X$, the representation $\pi_x = \pi \otimes e_x$ in the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{e_x} \mathbb{C}$ factors through a representation of A_x. Furthermore, if π_x is a faithful representation of A_x for every $x \in X$, π is said to be a field of faithful representations of A.

A continuous field of states on A is a $C(X)$-linear map $\varphi : A \to C(X)$ such that for any $x \in X$, the map $\varphi_x = e_x \circ \varphi$ defines a state on A_x.

If π is a $C(X)$-representation of the $C(X)$-algebra A, the map $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous since $\langle \xi, \pi(a)\eta \rangle \in C(X)$ for every $\xi, \eta \in \mathcal{E}$. Therefore, if A admits a field of faithful representations π, the map $x \mapsto \|a_x\| = \|\pi_x(a)\|$ is continuous for every $a \in A$, which means that A is a continuous field of C*-algebras over X ([4]).

The converse is also true ([3] théorème 3.3): given a separable $C(X)$-algebra A, the following assertions are equivalent:

1. A is a continuous field of C*-algebras over X,
2. the map $p : \mathcal{S}_X(A) \to X$ is open,
3. A admits a field of faithful representations.

2 C*-norms on $(A \otimes_{alg} B)/\mathcal{J}(A, B)$

DEFINITION 2.1 *Given two $C(X)$-algebras A and B, we define the involutive ideal $\mathcal{J}(A, B)$ of the algebraic tensor product $A \otimes_{alg} B$ of elements $\alpha \in A \otimes_{alg} B$ such that $\alpha_x = 0$ in $A_x \otimes_{alg} B_x$ for every $x \in X$.*

By construction, the ideal $\mathcal{I}(A, B)$ is included in $\mathcal{J}(A, B)$.

PROPOSITION 2.2 *Assume that $\| \ \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A \otimes_{alg} B$ of two $C(X)$-algebras A and B.*

If $\| \ \|_\beta$ is zero on the ideal $\mathcal{I}(A, B)$, then

$$\|\alpha\|_\beta = 0 \text{ for all } \alpha \in \mathcal{J}(A, B).$$

Proof: Let D_β be the Hausdorff completion of $A \otimes_{alg} B$ for $\| \ \|_\beta$. By construction, D_β is a quotient of $A \otimes_{\max} B$. Furthermore, if C_Δ is the ideal of $C(X \times X)$ of functions which are zero on the diagonal, the image of C_Δ in $M(D_\beta)$ is zero.

As a consequence, the map from $A \otimes_{\max} B$ onto D_β factors through the quotient $A \overset{M}{\otimes}_{C(X)} B$ of $A \otimes_{\max} B$ by $C_\Delta \times (A \otimes_{\max} B)$.

But an easy diagram-chasing argument shows that $(A \overset{M}{\otimes}_{C(X)} B)_x = A_x \otimes_{\max} B_x$ for every $x \in X$ ([3] corollaire 3.17) and therefore the image of $\mathcal{J}(A, B) \subset A \otimes_{\max} B$ in $A \overset{M}{\otimes}_{C(X)} B$ is zero. $\square$

2.1 The maximal C*-norm

DEFINITION 2.3 *Given two $C(X)$-algebras A_1 and A_2, we denote by $\| \ \|_M$ the C*-semi-norm on $A_1 \otimes_{alg} A_2$ defined for $\alpha \in A_1 \otimes_{alg} A_2$ by*

$$\|\alpha\|_M = \sup\{\|(\sigma_1^x \otimes_{\max} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$.

As $\| \ \|_M$ is zero on the ideal $\mathcal{J}(A_1, A_2)$, if we identify $\| \ \|_M$ with the C*-semi-norm induced on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$, we get:

PROPOSITION 2.4 *The semi-norm $\| \|_M$ is the maximal* C*-*norm on the quotient* $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Proof: By construction, $\| \|_M$ defines a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Moreover, as the quotient $A_1 \overset{M}{\otimes}_{C(X)} A_2$ of $A_1 \otimes_{\max} A_2$ by $C_\Delta \times (A_1 \otimes_{\max} A_2)$ maps injectively in

$$\underset{x \in X}{\oplus} (A_1 \overset{M}{\otimes}_{C(X)} A_2)_x = \underset{x \in X}{\oplus} ((A_1)_x \otimes_{\max} (A_2)_x),$$

the norm of $A_1 \overset{M}{\otimes}_{C(X)} A_2$ coincides on the dense subalgebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ with $\| \|_M$. But we saw in proposition 2.2 that if $\| \|_\beta$ is a C*-norm on the algebra $(A \otimes_{alg} B)/\mathcal{J}(A, B)$, the completion of $(A \otimes_{alg} B)/\mathcal{J}(A, B)$ for $\| \|_\beta$ is a quotient of $A \overset{M}{\otimes}_{C(X)} B$. $\square$

2.2 The minimal C*-norm

DEFINITION 2.5 *Given two $C(X)$-algebras A_1 and A_2, we define the semi-norm $\| \|_m$ on $A_1 \otimes_{alg} A_2$ by the formula*

$$\|\alpha\|_m = \sup\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|, x \in X\}$$

where σ_i^x is the map $A_i \to (A_i)_x$ and we denote by $A_1 \overset{m}{\otimes}_{C(X)} A_2$ the Hausdorff completion of $A_1 \otimes_{alg} A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1 \overset{m}{\otimes}_{C(X)} A_2)_x \to (A_1)_x \otimes_{\min} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\| \|_m$ induces a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. We are going to prove that this C*-norm defines the minimal C*-norm on the involutive algebra $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Let us introduce some notation.

Given two unital $C(X)$-algebras A_1 and A_2, let $P(A_i) \subset \mathcal{S}_X(A_i)$ denote the set of pure states on A_i and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples (ω_1, ω_2) such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_i) \to X$ is the restriction to $P(A_i)$ of the map $p : \mathcal{S}_X(A_i) \to X$ defined in section 1.

LEMMA 2.6 *Assume that $\| \|_\beta$ is a* C*-*semi-norm on the algebraic tensor product $A_1 \otimes_{alg} A_2$ of two unital $C(X)$-algebras A_1 and A_2 which is zero on the ideal $\mathcal{J}(A_1, A_2)$ and define the closed subset $S_\beta \subset P(A_1) \times_X P(A_2)$ of couples (ω_1, ω_2) such that*

$$\Big|(\omega_1 \otimes \omega_2)(\alpha)\Big| \leq \|\alpha\|_\beta \text{ for all } \alpha \in A_1 \otimes_{alg} A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Proof: Define for $i = 1, 2$ the adjoint action ad of the unitary group $\mathcal{U}(A_i)$ of A_i on the pure states space $P(A_i)$ by the formula

$$[(ad_u)\omega](a) = \omega(u^*au).$$

Then S_β is invariant under the product action $ad \times ad$ of $\mathcal{U}(A_1) \times \mathcal{U}(A_2)$ and we can therefore find non empty open subsets $U_i \subset P(A_i)$ which are invariant under the action of $\mathcal{U}(A_i)$ such that $(U_1 \times U_2) \cap S_\beta = \emptyset$.

Now, if K_i is the complement of U_i in $P(A_i)$, the set

$$K_i^\perp = \{a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i\}$$

is a non empty ideal of A_i and furthermore, if $\omega \in P(A_i)$ is zero on $K_i^\perp$, then ω belongs to K_i ([8] lemma 8,[15]).

As a consequence, if (φ_1, φ_2) is a point of $U_1 \times_X U_2$, there exist non zero self-adjoint elements $a_i \in K_i^\perp$ such that $\varphi_i(a_i) = 1$. If $x = p(\varphi_i)$, this implies in particular that $(a_1)_x \otimes (a_2)_x \neq 0$, and hence $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$. $\square$

LEMMA 2.7 *([15] theorem 1) Let A_1 and A_2 be two unital $C(X)$-algebras.*

If the algebra A_1 is an abelian algebra, there exists only one C**-norm on the quotient $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.*

Proof: Let $\| \ \|_\beta$ be a C*-semi-norm on $A_1 \otimes_{alg} A_2$ such that for all $\alpha \in A_1 \otimes_{alg} A_2$, $\|\alpha\|_\beta = 0$ if and only if $\alpha \in \mathcal{J}(A_1, A_2)$.

If $\rho \in P(A_\beta)$ is a pure state on the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for the semi-norm $\| \ \|_\beta$, then for every $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$,

$$\rho(a_1 \otimes a_2) = \rho(a_1 \otimes 1)\rho(1 \otimes a_2)$$

since $A_1 \otimes 1$ is included in the center of $M(A_\beta)$. Moreover, if we define the states ω_1 and ω_2 by the formulas $\omega_1(a_1) = \rho(a_1 \otimes 1)$ and $\omega_2(a_2) = \rho(1 \otimes a_2)$, then ω_2 is pure since ρ is pure, and $(\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)$. It follows that $P(A_\beta)$ is isomorphic to S_β.

In particular, if $a_1 \otimes a_2 \in A_1 \otimes_{alg} A_2$ verifies

$$(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \text{ for all couples } (\omega_1, \omega_2) \in S_\beta,$$

the element $a_1 \otimes a_2$ is zero in A_β and therefore belongs to the ideal $\mathcal{J}(A_1, A_2)$. Accordingly, the previous lemma implies that $P(A_1) \times_X P(A_2) = S_\beta = P(A_\beta)$.

As a consequence, we get for every $\alpha \in A_1 \otimes_{alg} A_2$

$$\begin{aligned} \|\alpha\|_\beta^2 &= \sup\{\rho(\alpha^*\alpha), \rho \in P(A_\beta)\} \\ &= \sup\{(\omega_1 \otimes \omega_2)(\alpha^*\alpha), (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)\} \end{aligned}$$

But that last expression does not depend on $\| \ \|_\beta$, and hence the unicity. $\square$

PROPOSITION 2.8 *([15] theorem 2) Let A_1 and A_2 be two unital $C(X)$-algebras.*

If $\| \ \|_\beta$ is a C**-semi-norm on $A_1 \otimes_{alg} A_2$ whose kernel is $\mathcal{J}(A_1, A_2)$, then*

$$\forall \alpha \in A_1 \otimes_{alg} A_2, \quad \|\alpha\|_\beta \geq \|\alpha\|_m.$$

Proof: If we show that $S_\beta = P(A_1) \times_X P(A_2)$, then for every $\rho \in S_\beta$ and every α in $A_1 \otimes_{alg} A_2$, we have $\rho(s^*\alpha^*\alpha s) \leq \rho(s^*s)\|\alpha\|_\beta^2$ for all $s \in A_1 \otimes_{alg} A_2$. Therefore

$$
\begin{aligned}
\|\alpha\|_m{}^2 &= \sup\Big\{\|(\sigma_1^x \otimes_{\min} \sigma_2^x)(\alpha)\|^2, x \in X\Big\} \\
&= \sup\Big\{\frac{(\omega_1 \otimes \omega_2)(s^*\alpha^*\alpha s)}{(\omega_1 \otimes \omega_2)(s^* s)}, (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2) \text{ and} \\
&\qquad\qquad s \in A_1 \otimes_{alg} A_2 \text{ such that } (\omega_1 \otimes \omega_2)(s^* s) \neq 0\Big\} \\
&\leq \|\alpha\|_\beta{}^2.
\end{aligned}
$$

Suppose that $S_\beta \neq P(A_1) \times_X P(A_2)$. Then there exist thanks to lemma 2.6 self-adjoint elements $a_i \in A_i$ and a point $x \in X$ such that $(a_1)_x \otimes (a_2)_x \neq 0$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Let B be the unital abelian $C(X)$-algebra generated by $C(X)$ and a_1 in A_1. The preceding lemma implies that $B\overset{m}{\otimes}_{C(X)}A_2$ maps injectively into the Hausdorff completion A_β of $A_1 \otimes_{alg} A_2$ for $\|\ \|_\beta$.

Consider pure states $\rho \in P(B_x)$ and $\omega_2 \in P(\,(A_2)_x)$ such that $\rho(a_1) \neq 0$ and $\omega_2(a_2) \neq 0$ and extend the pure state $\rho \otimes \omega_2$ on $B\overset{m}{\otimes}_{C(X)}A_2$ to a pure state ω on A_β. If we set $\omega_1(a) = \omega(a \otimes 1)$ for $a \in A_1$, then ω_1 is pure and $\omega(\alpha) = (\omega_1 \otimes \omega_2)(\alpha)$ for all $\alpha \in A_1 \otimes_{alg} A_2$ since ω and ω_2 are pure ([15] lemma 4). As a consequence, $(\omega_1, \omega_2) \in S_\beta$, which is absurd since $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = \rho(a_1)\omega_2(a_2) \neq 0$. $\square$

PROPOSITION 2.9 *Given two $C(X)$-algebras A_1 and A_2, the semi-norm $\|\ \|_m$ defines the minimal* C*-*norm on the involutive algebra* $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$.

Proof: Let $\|\ \|_\beta$ be a C*-norm on $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$. Thanks to the previous proposition, all we need to prove is that one can extend $\|\ \|_\beta$ to a C*-norm on $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the unital $C(X)$-algebras associated to the $C(X)$-algebras A_1 and A_2 (definition 1.1).

Consider the Hausdorff completion D_β of $(A_1 \otimes_{alg} A_2)/\mathcal{J}(A_1, A_2)$ and denote by π_i the canonical representation of A_i in $M(D_\beta)$ for $i = 1, 2$. Let us define the representation $\widetilde{\pi}_i$ of $\mathcal{A}_i$ in $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ by the following formulas:

$$
\begin{aligned}
\widetilde{\pi}_1(b_1 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_1(b_1) + g)\alpha \oplus (b_1 + f)a_1 \oplus f a_2 \oplus fg \\
\widetilde{\pi}_2(b_2 + u(f))(\alpha \oplus a_1 \oplus a_2 \oplus g) &= (\pi_2(b_2) + g)\alpha \oplus f a_1 \oplus (b_2 + f)a_2 \oplus fg
\end{aligned}
$$

For $i = 1, 2$, let $\varepsilon_i : \mathcal{A}_i \to C(X)$ be the map defined by

$$\varepsilon_i[a + u(f)] = f \text{ for } a \in A_i \text{ and } f \in C(X).$$

Then using the maps $(\varepsilon_1 \otimes \varepsilon_2)$, $(\varepsilon_1 \otimes id)$ and $(id \otimes \varepsilon_2)$, one proves easily that if $\alpha \in \mathcal{A}_1 \otimes_{alg} \mathcal{A}_2$, $(\widetilde{\pi}_1 \otimes \widetilde{\pi}_2)(\alpha) = 0$ if and only if α belongs to $\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$.

Therefore, the norm of $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ restricted to the subalgebra $(\mathcal{A}_1 \otimes_{alg} \mathcal{A}_2)/\mathcal{J}(\mathcal{A}_1, \mathcal{A}_2)$ extends $\|\ \|_\beta$. $\square$

Remark: As the $C(X)$-algebra A is nuclear if and only if every fibre A_x is nuclear ([12]), $A\overset{M}{\otimes}_{C(X)}B \simeq A\overset{m}{\otimes}_{C(X)}B$ for every $C(X)$-algebra B if and only if A is nuclear.

3 When does the equality $\mathcal{I}(A,B)=\mathcal{J}(A,B)$ hold?

Given two $C(X)$-algebras A and B, Giordano and Mingo have studied in [6] the case where the algebra $C(X)$ is a von Neumann algebra: their theorem 3.1 and lemma 1.5 of [10] imply that in that case, we always have the equality $\mathcal{I}(A,B)=\mathcal{J}(A,B)$.

Our purpose in this section is to find sufficient conditions on the $C(X)$-algebras A and B in order to ensure this equality and to present a counter-example in the general case.

PROPOSITION 3.1 *Let X be a second countable Hausdorff compact space and let A and B be two $C(X)$-algebras.*

If A is a continuous field of C*-*algebras over X, then $\mathcal{I}(A,B)=\mathcal{J}(A,B)$.*

Proof: Let us prove by induction on the non negative integer n that if

$$s=\sum_{1\le i\le n} a_i\otimes b_i\in\mathcal{J}(A,B),$$

then s belongs to the ideal $\mathcal{I}(A,B)$.

If $n=0$, there is nothing to prove. Consider therefore an integer $n>0$ and suppose the result has been proved for any $p<n$.

Fix an element $s=\sum\limits_{1\le i\le n} a_i\otimes b_i\in\mathcal{J}(A,B)$ and define the continuous positive function $h\in C(X)$ by the formula $h(x)^{10}=\sum\|(a_k)_x\|^2$.

The element $a'_k=h^{-4}a_k$ is then well defined in A for every k since $a_k^*a_k\le h^{10}$. Consequently, the function $f_k(x)=\|(a'_k)_x\|$ is continuous.

For $1\le k\le n$, let D_k denote the separable $C(X)$-algebra generated by 1 and the ${a'_k}^*a'_j$, $1\le j\le n$, in the unital $C(X)$-algebra $\mathcal{A}$ associated to A (definition 1.1). Then D_k is a unital continuous field of C*-algebras over X (see for instance [3] proposition 3.2).

Consider the open subset $\mathcal{S}^k=\{\psi\in\mathcal{S}_X(D_k)/\,\psi[{a'_k}^*a'_k]>\psi(f_k^2/2)\}$. If we apply lemma 3.6 *b)* of [3] to the restriction of $p:\mathcal{S}_X(D_k)\to X$ to $\mathcal{S}^k$, we may construct a continuous field of states ω_k on D_k such that $\omega_k[{a'_k}^*a'_k]\ge f_k^2/2$.

Now, if we set $s'=\sum_i a'_i\otimes b_i$, as $({a'_k}^*\otimes 1)s'$ belongs to $\mathcal{J}(D_k,B)$,

$$(\omega_k\otimes id)[({a'_k}^*\otimes 1)s']=\omega_k[{a'_k}^*a'_k]b_k+\sum_{j\ne k}\omega_k[{a'_k}^*a'_j]b_j=0.$$

Noticing that f_k^3 is in the ideal of $C(X)$ generated by $\omega_k[{a'_k}^*a'_k]$, we get that $f_k^3b_k$ belongs to the $C(X)$-module generated by the b_j, $j\ne k$, and thanks to the induction hypothesis, it follows that $(f_k^3\otimes 1)s'\in\mathcal{I}(A,B)$ for each k.

But $h^2=\sum_k f_k^2$ and so $h^4\le n\sum_k f_k^4$ is in the ideal of $C(X)$ generated by the f_k^3, which implies $s=(h^4\otimes 1)s'\in\mathcal{I}(A,B)$. $\square$

Remarks: **1.** As a matter of fact, it is not necessary to assume that the space X satisfies the second axiom of countability thanks to the following lemma of [11]: if $P(a)\in C(X)$

denotes the map $x \mapsto \|a_x\|$ for $a \in \mathcal{A}$, there exists a separable C*-subalgebra $C(Y)$ of $C(X)$ with same unit such that if D_k is the separable unital C*-algebra generated by $C(Y).1 \subset \mathcal{A}$ and the $a_k'^* a_j'$, $1 \leq j \leq n$, then $P(D_k) = C(Y)$. Furthermore, if $\Phi : X \to Y$ is the transpose of the inclusion map $C(Y) \hookrightarrow C(X)$ restricted to pure states, the map $D/(C_{\Phi(x)}(Y)D) \to A_x$ is a monomorphism for every $x \in X$ since $\mathcal{A}$ is continuous.

Consider now a continuous field of states $\omega_k : D_k \to C(Y)$ on the continuous field D_k over Y; if $\sum c^j \otimes d^j \in D_k \otimes_{alg} B$ is zero in $A_x \otimes_{alg} B_x$ for $x \in X$, then $\sum \omega_k(c^j)(x) d^j_x = 0$ in B_x, which enables us to conclude as in the separable case.

2. J. Mingo has drawn the author's attention to the following result of Glimm ([7] lemma 10): if $C(X)$ is a von Neumann algebra and A is a $C(X)$-algebras, then $P(a)^2 = \min\{z \in C(X)_+,\ z \geq a^*a\}$ is continuous for every $a \in \mathcal{A}$. Therefore we always have in that case the equality $\mathcal{I}(A,B) = \mathcal{J}(A,B)$ thanks to the previous remark.

COROLLARY 3.2 *Let A and B be two $C(X)$-algebras and assume that there exists a finite subset $F = \{x_1, \cdots, x_p\} \subset X$ such that for all $a \in A$, the function $x \mapsto \|a_x\|$ is continuous on $X \backslash F$.*

Then the ideals $\mathcal{I}(A,B)$ and $\mathcal{J}(A,B)$ coincide.

Proof: Fix an element $\alpha = \sum_{1 \leq i \leq n} a_i \otimes b_i \in A \otimes_{alg} B$ which belongs to $\mathcal{J}(A,B)$. In particular, we have $\sum_i (a_i)_x \otimes (b_i)_x = 0$ in $A_x \otimes_{alg} B_x$ for each $x \in F$.

As a consequence, thanks to theorem III of [13], we may find complex matrices $(\lambda^m_{i,j})_{i,j} \in M_n(\mathbb{C})$ for all $1 \leq m \leq p$ such that, if we define the elements $c^m_k \in A$ and $d^m_k \in B$ by the formulas

$$c^m_k = \textstyle\sum_i \lambda^m_{i,k} a_i \text{ and } d^m_k = b_k - \sum_j \lambda^m_{k,j} b_j,$$

we have $(c^m_k)_{x_m} = 0$ and $(d^m_k)_{x_m} = 0$ for all k and all m.

Consider now a partition $\{f_l\}_{1 \leq l \leq p}$ of $1 \in C(X)$ such that for all $1 \leq l, m \leq p$, $f_l(x_m) = \delta_{l,m}$ where δ is the Kronecker symbol and define for all $1 \leq k \leq n$ the elements $c_k = \sum_m f_m c^m_k$ and $d_k = \sum_m f_m d^m_k$. Thus,

$$\begin{aligned} \alpha &= \left(\textstyle\sum_i a_i \otimes d_i\right) + \left(\textstyle\sum_{i,j,m} \lambda^m_{i,j} a_i \otimes f_m b_j\right) \\ &= \left(\textstyle\sum_i a_i \otimes d_i\right) + \left(\textstyle\sum_j c_j \otimes b_j\right) + \left(\textstyle\sum_{i,j,m} \lambda^m_{i,j} (a_i \otimes f_m b_j - f_m a_i \otimes b_j)\right) \end{aligned}$$

and there exists therefore an element $\beta \in \mathcal{I}(A,B)$ such that $\alpha - \beta$ admits a finite decomposition $\sum_i a'_i \otimes b'_i$ with $a'_i \in C_0(X \backslash F)A$ and $b'_i \in C_0(X \backslash F)B$.

But $C_0(X \backslash F)A$ is a continuous field. Accordingly, proposition 3.1 implies that $\alpha - \beta \in \mathcal{I}(C_0(X \backslash F)A, C_0(X \backslash F)B) \subset \mathcal{I}(A,B)$. $\square$

Remark: If $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$ and if A and B are two $C(\widetilde{\mathbb{N}})$-algebras, the corollary 3.2 implies the equality $\mathcal{I}(A,B) = \mathcal{J}(A,B)$.

Let us now introduce a counter-example in the general case.

Consider a dense countable subset $X = \{a_n\}_{n \in \mathbb{N}}$ of the interval $[0,1]$. The C*-algebra $C_0(\mathbb{N})$ of sequences with values in $\mathbb{C}$ vanishing at infinity is then endowed with the $C([0,1])$-algebra structure defined by:

$$\forall f \in C([0,1]),\ \forall \alpha = (\alpha_n) \in C_0(\mathbb{N}), \quad (f.\alpha)_n = f(a_n)\alpha_n \text{ for } n \in \mathbb{N}.$$

If we call A this $C([0,1])$-algebra, then $A_x = 0$ for all $x \notin X$.

Indeed, assume that $x \notin X$ and take $\alpha \in A$. If $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\alpha_n| < \varepsilon$ for all $n \geq N$. Consider a continuous function $f \in C([0,1])$ such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(a_i) = 1$ for every $1 \leq i \leq N$; we then have:

$$\|\alpha_x\| \leq \|(1-f)\alpha\| < \varepsilon.$$

Let $Y = \{b_n\}_{n\in\mathbb{N}}$ be another dense countable subset of $[0,1]$ and denote by B the associated $C([0,1])$-algebra whose underlying algebra is $C_0(\mathbb{N})$.

Then, if $X \cap Y = \emptyset$, the previous remark implies that for every $x \in [0,1]$, $A_x \otimes_{alg} B_x = 0$ and hence, $\mathcal{J}(A,B) = A \otimes_{alg} B$. What we therefore need to prove is that the ideal $\mathcal{I}(A,B)$ is strictly included in $A \otimes_{alg} B$.

Let us fix two sequences $\alpha \in A$ and $\beta \in B$ whose terms are all non zero and suppose that $\alpha \otimes \beta$ admits a decomposition $\sum_{1\leq i\leq k}[(f_i\alpha^i) \otimes \beta^i - \alpha^i \otimes (f_i\beta^i)]$ in $A \otimes_{alg} B$. Then for every $n, m \in \mathbb{N}$,

$$\alpha_n\beta_m = \sum_{1\leq i\leq k} \alpha_n^i\beta_m^i[f_i(a_n) - f_i(b_m)].$$

Now, if we set $\phi_i(a_n) = \alpha_n^i/\alpha_n$ and $\psi_i(b_n) = \beta_n^i/\beta_n$ for $1 \leq i \leq k$ and $n \in \mathbb{N}$, this equality means that for all $(x,y) \in X \times Y$,

$$1 = \textstyle\sum_{1\leq i\leq k}\psi_i(x)\varphi_i(y)[f_i(x) - f_i(y)].$$

But this is impossible because of the following proposition:

PROPOSITION 3.3 *Let X and Y be two dense subsets of the interval $[0,1]$ and let n be a non negative integer.*

Given continuous functions f_i on $[0,1]$ and numerical functions $\psi_i : X \to \mathbb{C}$ and $\varphi_i : Y \to \mathbb{C}$ for $1 \leq i \leq n$, if there exists a constant $c \in \mathbb{C}$ such that

$$\forall (x,y) \in X \times Y, \qquad \sum_{1\leq i\leq n} \psi_i(x)\varphi_i(y)[f_i(x) - f_i(y)] = c$$

then $c = 0$.

Proof: We shall prove the proposition by induction on n.

If $n = 0$, the result is trivial. Take therefore $n > 0$ and assume that the proposition is true for any $k < n$.

Suppose then that the subsets X and Y of $[0,1]$, the functions f_i, ψ_i and φ_i, $1 \leq i \leq n$, satisfy the hypothesis of the proposition for the constant c.

For $x \in X$, let $p(x) \leq n$ be the dimension of the vector space generated in $\mathbb{C}^n$ by the $\big(\varphi_i(y)[f_i(x) - f_i(y)]\big)_{1\leq i\leq n}$, $y \in Y$.

If $p(x) < n$, there exists a subset $F(x) \subset \{1, \ldots, n\}$ of cardinal $p(x)$ such that for every $j \notin F(x)$:

$$\varphi_j(y)[f_j(x)-f_j(y)] = \sum_{i\in F(x)} \lambda_i^j(x)\varphi_i(y)[f_i(x)-f_i(y)] \text{ for all } y\in Y,$$

where the $\lambda_i^j(x)\in\mathbb{C}$ are given by the Cramer formulas. As a consequence,

$$\sum_{i\in F(x)}\Big(\psi_i(x)+\sum_{j\notin F(x)}[\lambda_i^j(x)\psi_j(x)]\Big)\varphi_i(y)[f_i(x)-f_j(y)] = c.$$

Now, if $p(x)<n$ for every $x\in X$, there exists a subset $F\subset\{1,\dots,n\}$ of cardinal $p<n$ such that the interior of the closure of the set of those x for which $F(x)=F$ is not empty and contains therefore a closed interval homeomorphic to $[0,1]$. The induction hypothesis for $k=p$ implies that $c=0$.

Assume on the other hand that $x_0\in X$ verifies $p(x_0)=n$. We may then find $y_1,\cdots y_n$ in Y such that if we set

$$a_{i,j}(x)=\varphi_i(y_j)[f_i(x)-f_i(y_j)],$$

the matrix $(a_{i,j}(x_0))$ is invertible. There exists therefore a closed connected neighborhood I of x_0 on which the matrix $(a_{i,j}(x))$ remains invertible.

But for each $1\le j\le n$, $\sum\limits_i a_{i,j}(x)\psi_i(x)=c$ and therefore the $\psi_i(x)$ extend by the Cramer formulas to continuous functions on the closed interval I.

For $y\in Y\cap I$, let $q(y)$ denote the dimension of the vector space generated in $\mathbb{C}^n$ by the $\Big(\psi_i(x)[f_i(x)-f_i(y)]\Big)_{1\le i\le n}$, $x\in X\cap I$.

If $q(y)<n$ for every y, then the induction hypothesis implies $c=0$. But if there exists y_0 such that $q(y_0)=n$, we may find an interval $J\subset I$ homeomorphic to $[0,1]$ on which the φ_i extend to continuous functions; evaluating the starting formula at a point $(x,x)\in J\times J$, we get $c=0$. $\square$

4 The associativity

Given three $C(X)$-algebras A_1, A_2 and A_3, we deduce from [3] corollaire 3.17:

$$[(A_1\overset{M}{\otimes}_{C(X)}A_2)\overset{M}{\otimes}_{C(X)}A_3]_x=(A_1\overset{M}{\otimes}_{C(X)}A_2)_x\otimes_{\max}(A_3)_x=(A_1)_x\otimes_{\max}(A_2)_x\otimes_{\max}(A_3)_x,$$

which implies the associativity of the tensor product $\cdot\overset{M}{\otimes}_{C(X)}\cdot$ over $C(X)$.

On the contrary, the minimal tensor product $\cdot\overset{m}{\otimes}_{C(X)}\cdot$ over $C(X)$ is not in general associative. Indeed, Kirchberg and Wassermann have shown in [11] that if $\widetilde{\mathbb{N}}=\mathbb{N}\cup\{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$, there exist separable continuous fields A and B such that

$$(A\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}B)_\infty\neq A_\infty\otimes_{\min}B_\infty.$$

If we now endow the C*-algebra $D=\mathbb{C}$ with the $C(\widetilde{\mathbb{N}})$-algebra structure defined by $f.a=f(\infty)a$, then for all $C(\widetilde{\mathbb{N}})$-algebra D', we have

$$[D\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}D']_n = \begin{cases} 0 & \text{if } n \text{ is finite,} \\ (D')_\infty & \text{if } n = \infty. \end{cases}$$

Therefore, $[(A\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}B)\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}D]_\infty \simeq (A\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}B)_\infty$ whereas $[A\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}(B\overset{m}{\otimes}_{C(\widetilde{\mathbb{N}})}D)]_\infty$ is isomorphic to $A_\infty \otimes_{\min} B_\infty$.

However, in the case of (separable) continuous fields, we can deduce the associativity of $\cdot\overset{m}{\otimes}_{C(X)}\cdot$ from the following proposition:

PROPOSITION 4.1 *Let A and B be two $C(X)$-algebras.*

Assume π is a field of faithful representations of A in the Hilbert $C(X)$-module $\mathcal{E}$, then the morphism $a \otimes b \to \pi(a) \otimes b$ induces a faithful $C(X)$-linear representation of $A\overset{m}{\otimes}_{C(X)}B$ in the Hilbert B-module $\mathcal{E} \otimes_{C(X)} B$.

Proof: Notice that for all $x \in X$, we have $(\mathcal{E} \otimes_{C(X)} B) \otimes_B B_x = \mathcal{E}_x \otimes B_x$.

Now, as B maps injectively in $B_d = \oplus_{x\in X} B_x$, $\mathcal{L}_B(\mathcal{E} \otimes_{C(X)} B)$ maps injectively in $\oplus_{x\in X}\mathcal{L}_{B_x}(\mathcal{E}_x \otimes B_x) \subset \mathcal{L}_{B_d}(\mathcal{E} \otimes_{C(X)} B \otimes_B B_d)$ and therefore if $\alpha \in A \otimes_{alg} B$, we have $\|(\pi \otimes id)(\alpha)\| = \sup\limits_{x\in X} \|(\pi_x \otimes id)(\alpha_x)\| = \|\alpha\|_m$. $\square$

Accordingly, if for $1 \leq i \leq 3$, A_i is a separable continuous field of C*-algebras over X which admits a field of faithful representations in the $C(X)$-module $\mathcal{E}_i$, the $C(X)$-representations of $(A_1\overset{m}{\otimes}_{C(X)}A_2)\overset{m}{\otimes}_{C(X)}A_3$ and $A_1\overset{m}{\otimes}_{C(X)}(A_2\overset{m}{\otimes}_{C(X)}A_3)$ in the Hilbert $C(X)$-module $(\mathcal{E}_1 \otimes_{C(X)} \mathcal{E}_2) \otimes_{C(X)} \mathcal{E}_3 = \mathcal{E}_1 \otimes_{C(X)} (\mathcal{E}_2 \otimes_{C(X)} \mathcal{E}_3)$ are faithful, and hence the maps $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to (A_1\overset{m}{\otimes}_{C(X)}A_2)\overset{m}{\otimes}_{C(X)}A_3$ and $A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to A_1\overset{m}{\otimes}_{C(X)}(A_2\overset{m}{\otimes}_{C(X)}A_3)$ have the same kernel.

References

[1] B. BLACKADAR K-theory for Operator Algebras, M.S.R.I. Publications **5**, Springer Verlag, New York (1986).

[2] E. BLANCHARD *Représentations de champs de* C*-*algèbres*, C.R.Acad.Sc. Paris **314** (1992), 911-914.

[3] E. BLANCHARD *Déformations de* C*-*algèbres de Hopf*, thèse de Doctorat (Paris VII, nov. 1993).

[4] J. DIXMIER Les C*-algèbres et leurs représentations, Gauthiers-Villars Paris (1969).

[5] G.A. ELLIOTT *Finite Projections in Tensor Products von Neumann Algebras*, Trans. Amer. Math. Soc. **212** (1975), 47–60.

[6] T. GIORDANO and J.A. MINGO *Tensor products of* C*-*algebras over abelian subalgebras*, *preprint*, Queen's University, April 1994.

[7] J. GLIMM *A Stone-Weierstrass theorem for* C**-algebras*, Ann. of Math. **72** (1960), 216–244.

[8] J. GLIMM *Type I* C**-algebras*, Ann. Math. **73** (1961), 572–612.

[9] G.G. KASPAROV *Hilbert C^*-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133–150.

[10] G.G. KASPAROV *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201.

[11] E. KIRCHBERG and S. WASSERMANN *Operations on continuous bundles of* C**-algebras*, *preprint* (preliminary version), September 1993.

[12] C. LANCE *Tensor product and nuclear C^*-algebras*, Operator Algebras and Applications, Proc. Symposia Pure Math **38** (1982) Part I, 379–399.

[13] F.J. MURRAY and J. von NEUMANN *On rings of Operators*, Ann. of Math. **37** (1936), 116–229.

[14] M.A. RIEFFEL *Continuous fields of C^*-algebras coming from group cocycles and actions*, Math. Ann. **283** (1989), 631–643.

[15] M. TAKESAKI *On the cross norm of the direct product of* C**-algebras*, Tôhoku Math J. **16** (1964), 111–122.

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